

Linear Operators That Strongly Preserve Commuting Pairs of Boolean Matrices*

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ABSTRACT

We characterize those linear operators T , on the class \mathcal{M} of square Boolean matrices (respectively, on the set \mathcal{S} of symmetric Boolean matrices) for which $T(X)$ commutes with $T(Y)$ if and only if X commutes with Y , for all X, Y in \mathcal{M} (respectively, in \mathcal{S}).

1. INTRODUCTION AND SUMMARY

Partly because of their association with nonnegative real matrices, Boolean matrices [(0, 1) matrices with the usual arithmetic, except $1 + 1 = 1$] have been the subject of research by many authors. In 1982, Kim [7] published a compendium of results on the theory and applications of Boolean matrices.

Often, parallels are sought for results known for field-valued matrices. See e.g. deCaen and Gregory [5], Rao and Rao [9, 10], Richman and Schneider [11], Beasley and Pullman [2, 3].

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The set of commuting pairs of matrices, \mathfrak{C} , is the set of (unordered) pairs of matrices (X, Y) such that $XY = YX$. The linear operator T is said to *strongly preserve* \mathfrak{C} (or T *strongly preserves commuting pairs*) when $T(X)T(Y) = T(Y)T(X)$ if and only if $XY = YX$.

In 1976 Watkins [13] proved that if $n \geq 4$, \mathcal{M} is the set of $n \times n$ matrices over an algebraically closed field of characteristic 0, and L is a nonsingular linear operator on \mathcal{M} which preserves commuting pairs, then there exists an invertible S in \mathcal{M} , a nonzero scalar c , and a linear functional f such that either $L(X) = cSXS^{-1} + f(X)I$ or $L(X) = cSX^tS^{-1} + f(X)I$, for all X in \mathcal{M} . In 1978, Beasley [1] extended this to the case $n = 3$. Also in [1], Beasley showed that the same characterization holds if $n \geq 3$ and L strongly preserves commuting pairs. The real symmetric and complex Hermitian cases were first investigated by Chan and Lim [4] in 1982; the same results were established as in the general case, with the exception that the invertible matrix must be orthogonal or unitary. Further extensions and generalizations to more general fields were obtained by Radjavi [8] and Choi, Jafarian, and Radjavi [6].

Evidently, the following operations strongly preserve the set of commuting pairs of matrices:

- (1) transposition ($X \rightarrow X^t$);
- (2) similarity ($X \rightarrow SXS^{-1}$ for some fixed invertible matrix S).

The operation $X \rightarrow X + f(X)I$ preserves commuting pairs of ring-valued matrices strongly when f is a linear functional. The corresponding operation $X \rightarrow X + I$ preserves commuting pairs of Boolean matrices, but not strongly (see the example below).

In Theorem 3.1 we show that the semigroup of linear operators strongly preserving commuting pairs of Boolean matrices is generated by transpositions and similarity. Theorem 4.1 obtains the same characterization of linear operators strongly preserving commuting pairs of symmetric Boolean matrices.

2. PRELIMINARIES

Algebraic operations on Boolean matrices and such notions as *linearity* and *invertibility* are defined as if the underlying scalars were in a field.

We let $\mathcal{M}_{m,n}$ denote the set of all $m \times n$ Boolean matrices. The $n \times n$ identity matrix I_n and the $m \times n$ zero matrix $O_{m,n}$ are defined as for a field. The $m \times n$ matrix all of whose entries are zero except its (i, j) th, which is 1, is denoted E_{ij} . We call E_{ij} a *cell*. We denote the $m \times n$ matrix all of whose

entries are 1 by $J_{m,n}$. We omit the subscripts on I , O , and J when they are implied by the context.

EXAMPLE. If A and B are $n \times n$ Boolean matrices, then $A + I$ commutes with $B + I$ whenever A commutes with B . On the other hand, when $n > 1$, then $E_{11} + I = I$, and hence it commutes with $J + I = J$ even though E_{11} does not commute with J . Therefore $X \rightarrow X + I$ preserves commuting pairs of Boolean matrices, but not strongly.

If A and B are in $\mathcal{M} = \mathcal{M}_{m,n}$, we say B dominates A (written $B \geq A$ or $A \leq B$) if $b_{ij} = 0$ implies $a_{ij} = 0$ for all i, j . This provides a reflexive, transitive relation on \mathcal{M} .

Linearity of transformations is defined as for vector spaces over fields. A linear transformation on \mathcal{M} is completely determined by its behavior on the set of cells.

The number of nonzero entries in a matrix is denoted $|A|$.

A matrix S having at least one nonzero off-diagonal entry is a *line matrix* if all its nonzero entries lie on a line (a row or column); so $1 \leq |S| \leq n$. If the nonzero entries in S are all in a row, we call S a *row matrix* and S' a *column matrix*. We use R_i (C_i) to denote the row matrix (column matrix) with all entries in the i th row (column) equal 1. We say that cells E and F are *collinear* if there is a line matrix L such that $L \geq E + F$.

When X and Y are in \mathcal{M} , we define $X \setminus Y$ to be the matrix Z such that $z_{ij} = 1$ if and only if $x_{ij} = 1$ and $y_{ij} = 0$. For example, the matrix in \mathcal{M} having all off-diagonal entries 1 and all diagonal entries 0 is denoted K_n , or just K when n is understood. Thus, $K = J \setminus I$.

A linear operator T on \mathcal{M} is said to be *nonsingular* if $T(X) = O$ implies that $X = O$. A nonsingular linear operator on \mathcal{M} need not be invertible. If U is any matrix whose first column has all entries 1, then $X \rightarrow XU$ is nonsingular but never invertible, unless $m = n = 1$. Similarly, a matrix A is said to be *nonsingular* if $Ax = 0$ implies that $x = 0$ (x a column vector). If A has a nonzero entry in each column, then A is nonsingular. Also, when $m = n$, the only invertible matrices are permutation matrices. Therefore, many nonsingular Boolean matrices are not invertible.

3. THE GENERAL CASE

In the next remarks and in Lemmas 3.1 through 3.5, we let $\mathcal{M} = \mathcal{M}_{n,n}$, and T be a linear operator on \mathcal{M} . Let $\mathcal{C}(A)$ denote the commutator semigroup of A , i.e. $\mathcal{C}(A) = \{X \in \mathcal{M} : XA = AX\}$. Then $\mathcal{C}(J)$ consists of O

and the matrices X such that both X and X^t are nonsingular (recall that J is the matrix of 1's).

LEMMA 3.1. *If $A \in \mathcal{C}(J)$ is nonzero and $B \geq A$, then $B \in \mathcal{C}(J)$.*

Proof. Notice that $A \in \mathcal{C}(J)$ if and only if both A and A^t are nonsingular, and A and A^t are nonsingular if and only if A has no zero row or column. Therefore, if $A \in \mathcal{C}(J)$ and $B \geq A$, then B has no zero row or column. Hence $B \in \mathcal{C}(J)$. ■

LEMMA 3.2. *If T strongly preserves $\mathcal{C}(J)$, then T is bijective on the set of cells in \mathcal{M} .*

Proof. First we show that T is nonsingular. We may assume that $n > 1$. If $T(E) = O$ for some cell E , let M be a minimal matrix in $\mathcal{C}(J)$ dominating E , that is, $|M| \leq |X|$ for all $X \in \mathcal{C}(J)$ with $E \leq X$. Such a matrix exists because $J \in \mathcal{C}(J)$. Moreover, $M \neq E$, as $E \notin \mathcal{C}(J)$ because E is singular. Then $T(M) = T(E + M \setminus E) = T(M \setminus E)$, contrary to the fact that $M \in \mathcal{C}(J)$ and $M \setminus E \notin \mathcal{C}(J)$.

Since \mathcal{M} is finite, there exists some integer $p > 0$ such that T^p is idempotent. Let $Q = T^p$. Then Q preserves both $\mathcal{C}(J)$ and $\mathcal{M} \setminus \mathcal{C}(J)$, and Q is nonsingular.

Suppose E and F are cells and $E \leq Q(F)$. If $E \neq F$, then there is a line matrix L such that $L \geq E$ and $L \not\geq F$. Let $N = J \setminus L$, then $N + E \in \mathcal{C}(J)$ but $N + F \notin \mathcal{C}(J)$, as the former has no zero row or column and the latter does. We have $Q(N + E) = Q(N) + Q(E) \leq Q(N + F)$. Since $N + E \in \mathcal{C}(J)$, we have $Q(N + E) \in \mathcal{C}(J)$, so Lemma 3.1 implies $Q(N + F) \in \mathcal{C}(J)$. But $N + F \notin \mathcal{C}(J)$. This contradicts the fact that Q preserves $\mathcal{M} \setminus \mathcal{C}(J)$. Thus $E = F$, and hence Q is the identity on the cells of \mathcal{M} . Therefore T is bijective on the cells of \mathcal{M} . ■

LEMMA 3.3. *If T strongly preserves $\mathcal{C}(J)$, then T preserves the set of line matrices.*

Proof. Suppose M is a line matrix. If E and F are noncollinear cells and $E + F \leq T(M)$, choose a permutation matrix $P \geq E + F$ and let $X = T^{-1}(P \setminus (E + F))$. Then $X + M$ has a zero row or column, so $X + M \notin \mathcal{C}(J)$, even though $T(X + M) \geq P \in \mathcal{C}(J)$. Hence $T(X + M) \in \mathcal{C}(J)$ by Lemma 3.1, contradicting the fact that T preserves $\mathcal{M} \setminus \mathcal{C}(J)$. ■

LEMMA 3.4. *If \mathbf{T} strongly preserves $\mathcal{C}(J)$ then either (a) \mathbf{T} maps row matrices to row matrices and column matrices to column matrices, or (b) \mathbf{T} maps row matrices to column matrices and column matrices to row matrices.*

Proof. According to Lemma 3.3, $\mathbf{T}(R_1)$ is a row matrix R_i or a column-matrix C_i , for some i . Suppose that $\mathbf{T}(R_1) = R_i$. Select j distinct from i , and suppose that the line matrix $\mathbf{T}(R_j)$ is the column matrix C_l . Then $|\mathbf{T}(R_1 + R_j)| = |R_i + C_l| < 2n$ while $|R_1 + R_j| = 2n$, contradicting the bijectivity of \mathbf{T} . Hence \mathbf{T} maps row matrices to row matrices and (a) holds. A similar argument establishes (b) when $\mathbf{T}(R_1)$ is a column matrix. ■

LEMMA 3.5. *If \mathbf{T} strongly preserves $\mathcal{C}(J)$, then there are permutation matrices P and Q such that either (i) $\mathbf{T}(X) = PXQ$ for all $X \in \mathcal{M}$ or (ii) $\mathbf{T}(X) = PX^tQ$ for all $X \in \mathcal{M}$.*

Proof. By Lemma 3.4, either (a) \mathbf{T} maps row matrices to row matrices and column matrices to column matrices, or (b) \mathbf{T} maps row matrices to column matrices and column matrices to row matrices. Since \mathbf{T} is bijective, no two lines can be mapped to the same line. Let P be the permutation matrix that corresponds to the mapping \mathbf{T} induces between the row indices and the row [column] indices, and let Q be the permutation matrix that corresponds to the mapping \mathbf{T} induces between the column indices and the column [row] indices, according as (a) [(b)] holds. Obviously (i) follows when (a) holds and (ii) follows when (b) holds. ■

THEOREM 3.1. *A linear operator \mathbf{T} on $\mathcal{M}_{n,n}$ strongly preserves commuting pairs if and only if there exists a permutation matrix P such that either (a) $\mathbf{T}(X) = PXP^t$ for all $X \in \mathcal{M}$ or (b) $\mathbf{T}(X) = PX^tP^t$ for all $X \in \mathcal{M}$.*

Proof. We only need to prove the necessity, and we may assume that $n > 1$. Suppose that \mathbf{T} strongly preserves commuting pairs. We first show that $\mathbf{T}(J) = J$. Suppose $\mathbf{T}(J) = R$, and choose L such that $\mathbf{T}(L) = R$ and $|L| \leq |A|$ for all A with $\mathbf{T}(A) = R$. Since $\mathbf{T}(L) = \mathbf{T}(J)$, $\mathcal{C}(L) = \mathcal{C}(J)$.

Suppose $l_{11} = 1$. Since $JP = PJ$ and so $PJP^t = J$ for any permutation matrix, P , we must have that $PLP^t = L$ for any permutation matrix P . Thus, $l_{ii} = 1$ for all $i = 1, \dots, n$. Now, L has a nonzero off-diagonal entry. Otherwise $L \leq I$ and thus $\mathcal{C}(I) = \mathcal{C}(J)$, which is impossible for $n > 1$. Without loss of generality we may assume that $l_{12} = 1$. For any permutation matrix P , $PL = LP$, because $\mathcal{C}(L) = \mathcal{C}(J)$. In particular, if P fixes the first row [column], it follows that the first row of L has all entries equal 1. Also,

$P_i L = L P_i$, where P_i is the permutation matrix that interchanges the first row [column] with the i th and fixes the rest, $2 \leq i \leq n$. It follows that $L = J$.

Now suppose $l_{11} = 0$. An argument similar to the above shows that $l_{ij} = 0$ if and only if $i = j$. Now J , and hence L , commutes with $I + E_{12}$. However, $L(I + E_{12})$ has $(2, 2)$ entry equal to 1, while $(I + E_{12})L$ has $(2, 2)$ entry equal to 0, a contradiction.

Thus $\mathbf{T}(J) = J$, and hence \mathbf{T} strongly preserves $\mathcal{C}(J)$. Let \mathcal{P}_n denote the set of $n \times n$ permutation matrices. Then if case (i) of Lemma 3.5 holds, we have $\mathbf{T}(\mathcal{P}_n) = \mathcal{P}_n$ and $\mathbf{T}(I) = PQ$. Then PQ commutes with every member of \mathcal{P}_n and hence $PQ = I$. A similar argument holds if case (ii) of Lemma 3.5 holds. ■

4. THE SYMMETRIC CASE

In this section we will investigate those operators on the set of symmetric Boolean matrices that strongly preserve commuting pairs. Let \mathcal{S} denote the set of all symmetric matrices in $\mathcal{M}_{n,n}$. We define a *digon matrix* to be the sum of a cell and its transpose. Notice that diagonal digon matrices are cells. A *star matrix* is the sum of a line matrix and its transpose. Clearly all digon matrices and all star matrices are symmetric. Let $\hat{\mathcal{C}}(A)$ denote the subsemi-group of $\mathcal{C}(A)$ which lies in \mathcal{S}_n , that is, $\hat{\mathcal{C}}(A)$ is the commutator of A in \mathcal{S} .

Here, $\hat{\mathcal{C}}(J)$ is the set of all symmetric nonsingular matrices together with O . In order to prove the three lemmas of this section, make the following replacements in the proofs of the first three lemmas in Section 3: replace "cell" with "digon matrix," replace "line matrix" with "star matrix," and replace "noncollinear cells" with "digon matrices not dominated by a star matrix."

In the following lemmas, \mathbf{T} is a linear operator on \mathcal{S} .

LEMMA 4.1. If $A \in \hat{\mathcal{C}}(J)$, $B \in \mathcal{S}$, and $B \geq A$, then $B \in \hat{\mathcal{C}}(J)$.

LEMMA 4.2. If \mathbf{T} strongly preserves $\hat{\mathcal{C}}(J)$, then \mathbf{T} is bijective on the set of digon matrices.

LEMMA 4.3. If \mathbf{T} strongly preserves $\hat{\mathcal{C}}(J)$, then \mathbf{T} preserves the set of star matrices.

THEOREM 4.1. *A linear operator T on \mathcal{S} strongly preserves commuting pairs if and only if there exists a permutation matrix P such that either (a) $T(X) = PXP^t$ for all $X \in \mathcal{S}$ or (b) $T(X) = PX^tP^t$ for all $X \in \mathcal{S}$.*

Proof. Let σ be the map of $\{1, \dots, n\}$ to itself defined by $\sigma(i) = j$ if and only if the maximal star matrix on row and column i is mapped to one on row and column j . By Lemma 4.2, σ is one-to-one, and hence onto. Let P be the permutation matrix corresponding to σ . ■

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